

# Modelling social change with hypercomplex numbers

## Overview

The aim is to model historical change, using quaternions and octonions to characterise the state of a society along various dimensions. The desired model is one of extreme abstraction. Since social systems exhibit empirical regularities (e.g. see the work on urban scaling), the hypothesis is that at a sufficient level of abstraction it may be possible to express their evolution with simple formulae.

The expected workshop output is not a “solution” as such but generation of ideas and avenues for further research.

This is a paper summary. A video summary and a more extended video treatment are available at the links below:

<https://www.youtube.com/watch?v=YdQzabPfScE>

[https://www.youtube.com/watch?v=8yxg\\_dQt5YM](https://www.youtube.com/watch?v=8yxg_dQt5YM)

Note 1. The model involves various assumptions, some of which may seem controversial. One is that social behaviour is describable by mathematics. Another is that the rules of social behaviour are the same at all times and in all places. Justification of each assumption would take much space. This paper only attempts to give a quick intuition for key points. Ultimately, the usefulness of the approach cannot be decided a priori but depends on where it leads, which is what it is desired to explore.

Note 2. The main discussion of this paper uses a dynamical systems approach. As indicated at the end, this is only to give a rough sketch based on a familiar formalism, and one of the topics for consideration is to identify alternative ways of representing the dynamics.

Note 3. We leave aside all moral evaluation, notwithstanding the emotional content of some of the terms to be used below. To talk of ‘social cohesion’ as an aspect of ‘vigour’, for example, does not imply a judgement that social cohesion is a ‘good thing’.

## Explanandum

The things to be explained are:

- There is overall, accelerating growth in human capabilities, yet different regions have advanced at different rates, while some regions seem to have stagnated or even gone backwards.
- There are cycles in human affairs—empires rise and fall, while economies exhibit booms and busts.

A key to this is the “phoenix principle”—i.e. the principle that it is necessary to destroy the old in order to create the new (to move an electronics factory from thermionic valves to transistors, you had to throw out old production lines and build new ones). This need to destroy things that currently work creates resistance to change. In particular, the more successful your existing way of doing things, the more risky it seems to change, whereas those who are failing are more ready to experiment. In this way, success breeds failure and failure breeds success.

## Model

What we are investigating is patterns of relationships. Relationships exist among social actors. A relationship between two actors means that one actor's behaviour influences the other actor's behaviour, so that their behaviour becomes somehow co-ordinated or complementary. If actors do not influence each other's behaviour, if they are on different planets say, then there is no relationship and no society for us to study.

Start with complex numbers.

Suppose the state of any society can be captured by just two variables, which we will call *scale* and *vigour*.

- Scale captures the intensity of social interaction, which is high in a big city or modern society, and low in a remote village or ancient society. It encapsulates both population (size and density) and technological level (since technology allows people to live at higher densities and also to move around and communicate with each other more easily).
- Vigour is more difficult to define. It is something understood intuitively by historians and more quantitatively by economists. They know when the Roman Empire / economy is doing well and when it is not. Here it will be understood to relate in some way to the strength of relationships between social actors. For example, the health of an economy relates to whether economic relationships are numerous, strong and active, or few, weak and quiescent.

These concepts will be unpacked and some of the difficulties revealed as we proceed. Improving the understanding of these concepts, particularly with the development of metrics, is part of the problem.

For now, scale and vigour are understood to have the same physical dimension, which is an inverse time. They can be understood as measures of social interaction per unit time.

Scale and vigour are also understood to be related to the phoenix principle in that a high degree of societal vigour, being an index of the success of current practices, creates high friction towards societal innovation and hence towards increases in scale (which increases through societal innovation). That is, vigour suppresses changes in scale, so that fluctuations in vigour and scale are out of phase.

Now make these variables the parts of a complex number, so that we represent the state of a society by the complex number,  $\underline{R}$ .

Assumption 1. Let us then suppose that the rate of change of a society's state is proportional to its current state. For example, the more technology a society already has, the faster its rate of technological evolution. We have:

$$\dot{\underline{R}} \sim \underline{R}$$

Our solution is

$$\underline{R} \sim e^{\underline{z}t}$$

If we write  $\underline{z} = \alpha + \underline{i}\omega$ , we have

$$\underline{R} \sim e^{\alpha t} (\cos \omega t + \underline{i} \sin \omega t)$$

What then motivates interest in the use of complex numbers is that, given nothing more than the relatively justifiable Assumption 1, we get two things that we are looking for: (1) our societal state exhibits the two types of historical change, i.e. accelerating growth and cyclical oscillations, and (2) the change in one part of state (real / imaginary) is out of phase with the change in the other part of state, thus expressing the phoenix principle (insofar as we connect these two parts with scale and vigour respectively).

As stated, there are some difficulties, which I will gloss over for now and come back to as representing areas where assistance is sought in the form of new ideas and alternative approaches.

For now, without getting too bogged down in what this all means, and simply noting that there is something interesting here even though we do not fully understand it yet, let us move on.

We had the concept of societal vigour, which we said is something widely understood, even if it is at an intuitive level and not properly formulated. Let us now make the step of saying that vigour has *precisely three components*: political vigour, economic vigour, and social (or cultural) vigour. While I believe there are good a priori reasons for favouring this widely used dimensionalisation of human affairs, they do not matter here. The objective is to find where the model leads and evaluate it by its results, not by a priori debate.

The three components of vigour can be termed *political integration* (a measure of a society's unity of purpose, achieved through effective government and contrasting with disagreement and conflict), *economic organisation* (a measure of the division of labour and commercial interchange, contrasting with self-sufficiency and limited trade) and *social cohesion* (a measure of shared values and goals, contrasting with selfishness and mutual distrust).

There are various points to note in interpreting these concepts, which I will again return to below, for now keeping a broad brush.

In splitting vigour into three components, we go from a complex number to a quaternion, where scale again corresponds to the real part while integration, organisation and cohesion correspond to the imaginary parts.

With the same

$$\underline{\dot{R}} \sim \underline{R}$$

where  $\underline{R}$  is now a quaternion, we will get something like

$$\underline{R} \sim e^{at} (\cos \omega t + \underline{i} \beta_1 \sin \omega t + \underline{j} \beta_2 \sin \omega t + \underline{k} \beta_3 \sin \omega t)$$

We can now study political, economic and social evolution separately, while retaining the features of the complex number model, i.e. with two types of historical change and a phase difference between scale and (the components of) vigour.

The next development of the model is to recognise that human affairs involve both the *reality* of a situation and *perceptions* about that situation. Typically, people cannot know the reality of the situation and instead act on their perception of it. For intuition, reality corresponds to the true value of a financial instrument like a share (which depends on imponderables like future performance), while perception corresponds to the price at which it is traded, i.e. what the value is believed to be.

Reality and perception are both important, and crucially they affect each other. Perception to some extent reflects reality, and reality is also affected by perception (e.g. if a company's share price is

high, i.e. it has a high perceived value, it becomes easier to raise capital, develop the business and increase the real value).

We therefore represent a society's state by a pair of quaternions,  $\underline{R}$  (the real state) and  $\underline{P}$  (the perceived state).

A proposed starting point for a set of equations describing how the real and perceived states change is as follows:

$$\dot{\underline{R}} \sim \underline{R} \underline{P} \text{ and } \dot{\underline{P}} \sim \underline{P} \underline{R}$$

The non-commutative nature of quaternions means that these expressions are not equal.

Whether this formulation is the best one is part of the problem for consideration. The reason for choosing this formulation is not only that it is one of the simplest imaginable, but also that preliminary investigation suggests it yields sociologically plausible relationships between the variables of scale, integration, organisation and cohesion. See the box below for illustration.

Motivation for adopting the equation  $\dot{\underline{R}} \sim \underline{R} \underline{P}$ .

We identify the four parts of the quaternion with scale, integration, organisation and cohesion in that order. Use S, I, O and C to represent these quantities, with subscript R and P for real and perceived. Consider the second part of the quaternion, equating the coefficients on the left and right hand sides of the above equation. By the rules of quaternion multiplication, we get

$$\dot{I}_R \sim S_R I_P + S_P I_R + O_R C_P - O_P C_R$$

$\dot{I}_R$ , the rate of increase of real integration, represents the tendency of government to increase in effectiveness, i.e. to become more successful at imposing its will on society. This is the *real* aspect of that quantity, i.e. it represents not the understanding of the citizens but whether in reality the government has the means to impose its will if tested.

From the right hand side, we see that there is a positive effect on this quantity from  $O_R$ , which represents the real organisation, i.e. the real economic vigour or real wealth of the society. This says a more prosperous society makes for a more effective government—a glance around the world suggests there is indeed some correlation, albeit not perfect because this is just one factor, as the equation shows. Note that it is real prosperity that is important, not the perception of prosperity.

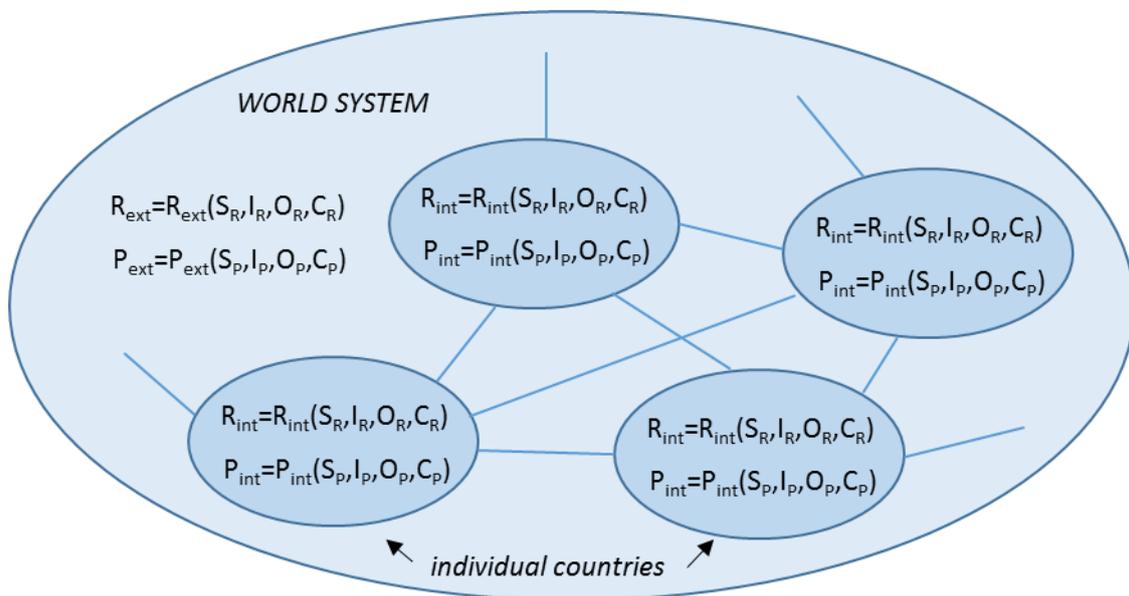
The positive effect of  $O_R$  is moderated by  $C_P$ , the perceived social cohesion. This expresses a common ethnographic observation, which is that governments find it hard to collect taxes where social trust is low. Real prosperity promotes real governmental effectiveness because the government participates in that prosperity through taxation, which funds effective mechanisms of government. However, the government's ability to participate in the prosperity depends on its ability to collect taxes, which depends on social cohesion. Note that it is perceived cohesion that is important here because this is about people's behaviour, which is based on beliefs.

This is to consider just one term in the expression for rate of change of just one sociological variable. There are eight such variables (four variables each with real and perceived versions) and four terms in the expression for each. Nevertheless, consideration of the other variables and terms leads to similarly sociologically plausible interpretations.

Note how the equation  $\dot{\underline{R}} \sim \underline{R} \underline{P}$  provides an extremely compact representation of a large number of sociological relationships.

The final step in development of the model is to recognise that social systems are nested. For example, we can talk about the political integration, economic organisation and social cohesion of a particular country, but that country is also subject to external diplomatic or military pressures (e.g. from the global hegemon, the United States), it has trading relationships with other countries, and it may be part of a regional bloc based on shared values and undertakings of mutual support. In other words, a country has internal political, economic and social networks, while itself being an actor in the global political, economic and social networks. The country's evolution depends on both the internal and the external dynamics.

Crudely, we might imagine a hierarchical model, with say global, national, metropolitan, corporate and individual levels. The evolution of each level would depend both on its internal scale, integration, organisation and cohesion (real and perceived) and on the external scale, integration, organisation and cohesion (real and perceived), i.e. the scale, integration, organisation and cohesion of the wider system of which it is a part. See the diagram below, where  $S_R$  = real scale,  $S_P$  = perceived scale,  $I_R$  = real integration etc.



There are certainly interactions between the levels: for example, the support of the United Nations can enhance the internal effectiveness or legitimacy of a government. We might be able to cope with this by moving from quaternions to octonions. Our real and perceived state variables, which are multiplied together, would now have eight components, corresponding to internal scale, integration, organisation and cohesion and external scale, integration, organisation and cohesion.

Meanwhile, with entities at one level being composed of entities at the next level down, what is internal at one level is external at the lower level. We should therefore construct a model that is self-similar and requires a recursive solution procedure.

The term 'crudely' was used above because in fact this idea of discrete levels (corresponding to real-world concepts like global, national, regional etc.) is insufficiently abstract for our purposes. In general, there are no preferred levels of analysis. An interaction between two countries is typically also an interaction (or set of interactions) between two corporations or between two individuals. How we interpret it (international, intercorporate or interpersonal) represents an arbitrary choice of reference frame. Therefore, we should be able to draw the boundaries of our units of analysis wherever we like and still get the same result. This is a crucial constraint on the model that should be important in finding solutions.

## Dimensions

Some points to note about the proposed dimensions (scale, integration, organisation and cohesion) are as follows:

- These terms should be understood abstractly; e.g. the division of labour involved in the organisation concept is any exchange of services and does not necessarily involve monetary transfers.
- We said that we are studying (patterns of) relationships, where relationships imply exertion of influence by one actor on another actor's behaviour. The three dimensions correspond to three modes of influence: *coercive* (integration), where an actor is forced to do something; *contractual* (organisation), where an actor does something in return for something else; *cooperative* (cohesion), where an actor does something spontaneously out of a sense of shared interest with the other actor.
- Any practical real-world institution involves all three modes of influence simultaneously. People pay their taxes partly because they will go to jail if they don't (coercive), partly because they get things like roads and schools in return (contractual), and partly because they consider it their civic responsibility (cooperative). We never see these things in pure form.
- Relationships are in one direction: one person influences another without the other person necessarily influencing them to the same extent or at all. Many relationships are in fact two-way, but these should be considered as composed of two distinct, directed relationships.
- To be at the influencing end of coercive relationships is to have power. To be at the influencing end of contractual relationships is to have wealth. To be at the influencing end of cooperative relationships is to have legitimacy.
- Integration, organisation and cohesion induce each other. For example, high integration, meaning effective government, implies a distinction between ruler and ruled, which corresponds to a division of labour, which constitutes organisation. Or, for rulers to rule, the population needs to have some sense of their legitimacy, i.e. there have to be shared values, i.e. cohesion. One could perhaps imagine a situation where every political decision is decided by a vote of all the citizens, implying integration without organisation, or a situation where a political authority rules entirely by fear, implying integration without cohesion, but probably not both together. Whether these interactions between the dimensions represent logical necessities or just practical tendencies is part of the problem for consideration.
- Given the presence of sines and cosines in the solution for the state variable, the various parts, i.e. scale, integration, organisation and cohesion, can take on negative values. We could possibly interpret negative values as follows. At one time, the Roman emperor ruled over a large area of Europe and North Africa, from Egypt to England and Wales. We can say that this constituted a political unit with a positive integration. As the effectiveness of the central Roman government declined, the integration of this area also declined but remained positive. Eventually, however, the empire broke up so that it was no longer a political unit but rather a multiplicity of political units. We could say that the integration of the original political unit had now become negative; it no longer had even minimal integration. Thus, we could say that, when the sine takes the value  $+1$ , it represents full integration, with the political system under the perfect control of one will (in this case that of the emperor); and when the sine takes the value  $-1$ , it represents complete fragmentation (disintegration) with each element of the former political system under the control of its own will.  $+1$  represents complete unity and  $-1$  represents complete disunity. Whether similar interpretations work for scale, organisation and cohesion, or whether we should somehow disallow negative values, is a question for consideration.

## Issues

Additional issues and questions for exploration are as follows.

- A fundamental issue is that of what we mean by scale and vigour, and how these relate to the parts of the complex number / quaternion. Do we equate scale and vigour (i.e. integration, organisation and cohesion) with the real and imaginary parts of the state variable, or should we be using more subtle, derived concepts of some kind?
  - If we do equate scale and vigour with the real and imaginary parts, we can understand each part as dividing into a relative component,  $\sin \omega t$  or  $\cos \omega t$ , which varies between  $-1$  and  $+1$ , and a multiplier,  $e^{\alpha t}$ , which allows indefinite growth and combines with the relative component to create the absolute value of scale or vigour.
  - The idea of relative and absolute components is easier to understand with respect to vigour. For example, in the case of integration we can say a high value means a well-ordered society governed by effective authorities, and a low value means a disorderly, weakly governed society characterised by crime, conflict and civil disobedience. In this respect, integration can be seen as a relative measure whereby we can fairly compare very different societies, saying for example that the Roman Republic had higher political effectiveness or higher integration than a modern failed state. Nevertheless, even very weak states today have access to far more capable technologies of political control, such as the telephone and computer records, than were available to the ancient Romans, and they manage larger, more complex populations, so that while the relative integration may be lower, the absolute integration is higher. The distinction between relative and absolute vigour is thus a meaningful one.
  - It is harder to understand what we mean by the relative and absolute components of scale. Perhaps what we called scale corresponds to just the multiplier,  $e^{\alpha t}$ , and it is something more subtle that comprises the real part of the quaternion. This quantity represented by the real part seems to have something to do with the intensity of interaction, but it is not clear whether it is the actual instantaneous intensity, which we called scale, or whether it represents the capacity for this intensity to change (through technological and institutional innovation).
- How do we measure these quantities of real and perceived scale, integration, organisation and cohesion?
- We said that scale, integration, organisation and cohesion have the same physical dimensions, of inverse time. This follows from the equation  $\dot{R} \sim R \underline{P}$  (so long as any constant of proportionality is dimensionless, that is). If we change the equation, it may change the quantities' dimensions, and therefore change how we conceptualise them and how we measure them.
- In the outline of the model above, integration, organisation and cohesion have been treated symmetrically. In reality, their role in human affairs does not seem to be symmetric. For one thing, they operate over different characteristic times. Coercive relationships operate over short times—if someone wants to get someone else do something, pointing a gun at them will likely get them to comply with little delay. Contractual relationships operate over intermediate times—to get someone to do something by offering them a payment or reward typically takes time, e.g. to understand their needs and negotiate a mutually acceptable bargain. Cooperative relationships operate over long timescales—to get someone to do something spontaneously out of their care and concern for one's wellbeing requires friendship, which can take a long time to cultivate. (Charitable acts towards strangers seem to contradict this but in fact are consistent with it when one considers all the factors such as the relative cost of the act and the degree of

control the recipient has over the other party's behaviour.) The question is therefore how this and possible other asymmetries should be incorporated into the model.

- The presentation of the model above used differential equations. This should be regarded as a rough sketch to give the basic idea. A more appropriate model may be one similar to that of earthquakes, where there is no sinusoidal oscillation but instead a build-up and release of stress in which the size of events has a scale-free distribution (as seems to be an empirical property of actual social systems).